THEORETICAL, COMPUTATIONAL AND EXPERIMENTAL INVESTIGATION OF HELMHOLTZ RESONATORS WITH FIXED VOLUME: LUMPED VERSUS DISTRIBUTED ANALYSIS†

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1. INTRODUCTION

The well-known Helmholtz resonator has found applications in a wide variety of technologically significant problems. Tang and Sirignano [1] provide an excellent discussion of the use of this resonator in reducing the organized oscillations inside jet engines, rocket combustors and furnaces. The concept of Helmholtz resonance and the associated classical theory have been applied in the design and analysis of various systems, including tuned intake manifolds of vehicles [2–5], tuned pulse combustors [6, 7], discharge systems of compressors and small engines [8, 9], as well as noise reduction elements [10–12]. Due to its inherent characteristics of attenuating low frequency noise, the Helmholtz resonator is used extensively in vehicle induction and exhaust systems. The present study focuses on this latter aspect and investigates the sound attenuation performance of the Helmholtz resonator.

Classical Helmholtz theory is commonly used in the design of the foregoing low frequency resonators to predict the resonance frequency and transmission loss characteristics. The theory applies Newton’s second law to a lumped mass in the resonator neck acting between an adiabatically compressed volume at one end and the forcing function (incident pressure on the resonator) at the other. For a Helmholtz resonator attached to an anechoically terminated duct with no mean flow and wavelengths much larger than any characteristic dimension, the lumped parameter model gives the resonance frequency, \( f_r \), and transmission loss, \( TL \), as

\[
    f_r = \frac{c_0}{2\pi} \sqrt{\frac{A_e}{l \cdot V}}, \quad TL = 10 \log_{10} \left[ 1 + \left( \frac{\sqrt{A_e V l}}{2 f f_r} \right)^2 \right],
\]

where \( c_0 \) is the mean speed of sound, \( A_e \) is the cross-sectional area of the connector (neck), \( l \) is the connector length, \( V \) is the resonator volume and \( A_p \) is the cross-sectional area of the main duct (pipe). Experimental observations, however, usually deviate from

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the theory due to the simplifications involved in reducing distributed, multi-dimensional phenomena to lumped parameters [10]. In addition, wave motion neglected by the classical approach may have a significant effect as the neck or volume lengths reach 5–10% of the wavelength [12–14]. These discrepancies are accounted for by adding a correction to the actual connector length, with the correction usually consisting of two separate components, one for each end of the neck. Various expressions have been suggested for the correction at the volume interface [11, 12, 15–18], while the correction at the main duct junction remains somewhat unclear, due primarily to the three-dimensional structure. Since the Helmholtz resonator is designed to attenuate a narrow frequency band, the uncertainty in these corrections readily leads to improper designs.

Equation (1) clearly shows that for a fixed neck geometry and constant volume, classical theory yields a single resonance frequency and transmission loss curve that are independent of the individual volume dimensions. In realistic Helmholtz resonator geometries, including the induction and exhaust systems of automotive engines, the cavity has definable dimensions which can lead to situations in which the fluid motion in this volume has significance.

The objective of the present study is to investigate the effect of changing the volume length-to-diameter ratio, \( \frac{L}{d} \), on the resonance frequency and transmission loss of concentric, cylindrical Helmholtz resonators with fixed reservoir volume; and to show the relevance of the results to the end correction and the design of effective resonators. Three different approaches are employed in the study: (1) a closed form solution for the transmission loss, obtained from linear acoustic theory, is presented which includes wave propagation along the axis of the neck and volume; (2) a computational time-domain technique is used to model Helmholtz resonators in an impedance tube—the finite difference code, based on the work of Chapman, Novak and Stein [19], has been developed to solve the one-dimensional, non-linear balance equations of mass, momentum and internal energy coupled with the equation of state; (3) six resonators of fixed neck geometry and constant volume have been constructed with \( \frac{L}{d} \) ratios ranging from 1.59 to 23.92—the transmission loss of these configurations was measured in an anechoically terminated impedance tube set-up using the two-microphone technique.

Section 2 discusses the analytical expression for transmission loss obtained from one-dimensional acoustic theory, including the behavior for varying volume dimensions, reduction to simpler relationships for some common geometries and comparison of resonance prediction with that obtained from other approaches. Section 3 introduces the computational method briefly with a subsequent presentation of the numerical predictions in section 4 for transmission loss of the six test resonators. Section 5 describes the experimental set-up, followed by the data obtained for the test resonators in section 6. Finally, section 7 concludes the study with some closing remarks.

2. CLOSED FORM ACOUSTIC ANALYSIS

The geometry considered in this study is shown inset in Figure 1, with fixed dimensions \( V \), \( d_c \), \( l \), and \( d_r \), and varied longitudinal and transverse volume dimensions, \( l \) and \( d \), respectively. Since the connector and main duct are both cylindrical, \( l_c \) is not clearly defined; therefore a mean of the extreme values, 8.5 cm, has been adopted for the analyses.

For planar propagation of sound waves, the transmission loss of an acoustic element is calculated as

\[
TL = 10 \log_{10} \left( \frac{C_{+r}}{C_{+l}} \right)^2,
\]  

(2)
where \( C_{+,i} \) and \( C_{+,ir} \) are complex constants representing magnitudes of the (harmonic) incident and transmitted pressure waves, respectively. Assuming constant pressures and conservation of volume flow at duct intersections, neglecting viscous effects and incorporating wave motion in the volume and neck, classical one-dimensional acoustic theory yields the Helmholtz resonator transmission loss as

\[
TL = 10 \log_{10} \left[ 1 + \frac{A_c}{2A_p} \left( 1 + \frac{\phi + (\phi - 1) e^{-2\pi k l_c}}{1 + \phi - (\phi - 1) e^{-2\pi k l_c}} \right)^2 \right],
\]

\[
\phi = \frac{A_v}{A_c} \left( \frac{e^{2\pi k l_c}}{e^{2\pi k l_c} - 1} \right),
\]

where \( A_v \) is the cross-sectional area of the volume and \( k = 2\pi / \lambda \) is the wavenumber, \( \lambda \) being the wavelength. Note that this expression and the computational results presented later are valid for planar wave propagation, which imposes the restrictions \( \pi d / \lambda \leq 1.841 \) for diametral modes and \( \pi d / \lambda \leq 3.832 \) for radial modes. Equation (3) may be rearranged to an equivalent trigonometric form as

\[
TL = 10 \log_{10} \left[ 1 + \left( \frac{A_c}{2A_p} \frac{\tan k l_c + (A_v/A_c) \tan kl}{1 - (A_v/A_c) \tan kl \tan k l_c} \right)^2 \right].
\]

This general expression may readily be shown to reduce to some well-known relationships for simpler geometries. For example, letting \( A_c = A_v \) results in

\[
TL = 10 \log_{10} \left[ 1 + \left( \frac{A_c}{2A_p} \tan k (l_c + l) \right)^2 \right],
\]

which represents, as expected, a quarter-wave resonator with branch length \( l_c + l \). Letting \( A_v/A_c \) approach zero gives an expression

\[
TL = 10 \log_{10} \left[ 1 + \left( \frac{A_c}{2A_p} \tan k l_c \right)^2 \right],
\]
for a side branch resonator of length \( l_c \). Provided that the constant volume constraint is not imposed, the limit as \( A_c/A_V \) approaches zero yields

\[
TL = 10 \log_{10} \left[ 1 + \left( \frac{A_c}{2A_p} \cot kl_c \right)^2 \right],
\]

(7)
corresponding to an open-ended side branch resonator. Additionally, if the volume is fixed and \( l \) approaches zero, equation (4) reduces to a lumped volume model which includes wave motion in the connector [14]. Finally, fixing the volume and letting both \( l \) and \( l_c \) approach zero, such that \( kl \ll 1 \) and \( kl_c \ll 1 \), gives the lumped parameter expression, equation (1), with the assumption that \( k, l_c = \sqrt{V_c/V} \) is negligible in comparison with \( 1/k, l_c \), where \( k = 2\pi f/c_0 \) and \( V_c = A_c l_c \).

The resonator transmission loss becomes infinite as the denominator in equation (4) approaches zero, yielding an expression for resonance locations as

\[
\tan k l, \tan k l_c = A_c/A_V.
\]

(8)
This useful and relatively simple equation for resonance frequency is consistent with the work of Tang and Sirignano [1]. In addition, for a short neck, \( k, l_c \ll 1 \) and \( \tan k, l_c \approx k, l_c \), thus equation (8) may be simplified as

\[
(A_V/A_c)(l_c/l)k l = \cot k l,
\]

(9)
matching the analysis by Panton and Miller [12] for a lumped mass in the connector and distributed mass in the volume. Furthermore, for a volume considerably shorter than the wavelength, \( k, l \ll 1 \) or \( \cot k, l \approx 1/k, l \), equation (9) reduces as expected to

\[
f_r = \frac{c_0}{2\pi} \sqrt{\frac{A_c}{l_c V}},
\]

(10)
which is the classical approach result for lumped parameters in both the connector and volume.

The theoretical transmission loss given by equation (4) at a number of convenient \( l/d \) ratios is shown in Figure 1. For a given volume, the primary resonance frequency decreases with increasing \( l/d \). In view of equation (1), this trend is similar to artificially increasing the connector length. It may then be suggested that, for certain geometries, the need for a length correction is partially due to wave propagation throughout the volume, in addition to the actual end effects at duct interfaces. With increasing \( l/d \), the geometry is in a transition between Helmholtz and side branch resonator configurations, as expected; note, for example, the second resonances for \( l/d = 10 \) and 15.

3. COMPUTATIONAL APPROACH

For one-dimensional compressible flow in ducts of variable cross-section, the balance equations of mass, momentum and internal energy may be expressed as

\[
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho AU) = 0,
\]

(11)
\[
\frac{\partial}{\partial t} (\rho AU) + \frac{\partial}{\partial x} (\rho AU^2) + \frac{\partial}{\partial x} (p A) - \tau_w \mathcal{P} = 0,
\]

(12)
\[
\frac{\partial}{\partial t} (\rho Ae) + \frac{\partial}{\partial x} (\rho AU e) + p \frac{\partial}{\partial x} (UA) - \tau_w \mathcal{P} U + q \mathcal{P} = 0,
\]

(13)
where $\rho$ is the density, $A$ is the cross-sectional area, $U$ is the velocity, $p$ is the pressure, $\tau_w$ is the wall shear stress, $P$ is the perimeter, $e$ is the specific internal energy, and $q$ is the wall heat flux. The ideal gas equation of state,

$$p = (\gamma - 1)p e,$$

(14)

where $\gamma$ is the ratio of specific heats, is used to relate the thermodynamic variables and close the system of equations. Equations (11)-(13) are discretized by employing the explicit finite difference method of Chapman, Novak and Stein [19] as discussed by Selamet, Dickey and Novak [20], which may be referenced for a more detailed description of the technique. The staggered mesh used in the discretization divides a duct into cells with vector quantities located at node points, and scalar quantities at cell mid-points. In order to match the assumptions of inviscid, adiabatic flow used in the derivation of equation (4), the program functions for $\tau_w$ and $q$ were set to zero. The results obtained from the application of this approach are discussed in the following section.

4. COMPUTATIONAL RESULTS

The six volume configurations used in the computational and experimental study are presented in Table 1. The lowest $l/d = 1.59$ is nearly within the range of an adiabatic compression model (note that $\lambda/c \approx 14$, whereas Panton and Miller suggest $\lambda/c \geq 16$); while at the maximum $l/d = 23.92$ the configuration is well into the length-controlled range. In the computational study, the neck and the reservoir were modeled as a single ubranch duct connected the main pipe. Since the numerics require piecewise continuity of duct cross-sectional area, a small transition length was needed for the expansion from neck to volume, equivalent to the computational grid size of the branch duct, $\Delta x$. An effect of increasing this dimension is artificially to increase the connector length, thereby reducing the predicted resonance frequency. To examine this behavior, simulations were performed for the test resonators with $\Delta x$ varied from 0.1 to 0.5 cm. The primary resonance frequencies obtained, divided by the first resonance frequency from equation (8) are shown in Figure 2. For $\Delta x \leq 0.3$ cm, the discrepancy between the model and one-dimensional theory is less than 5% for all configurations, with the agreement improving as cell size is decreased and $l/d$ is increased. As a compromise between numerical accuracy and computation time, a value of $\Delta x = 0.3$ cm was adopted for all the simulations. Computations with $0.5 < \Delta x < 3.0$ cm exhibited a continuing decrease in the primary resonance frequency as $\Delta x$ increased.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Varied geometric parameters for the study (geometry of Figure 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Resonator</th>
<th>$l$ (cm)</th>
<th>$d$ (cm)</th>
<th>$l/d$</th>
<th>$d/d_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>24.420</td>
<td>15.319</td>
<td>1.59</td>
<td>3.79</td>
</tr>
<tr>
<td>B</td>
<td>35.281</td>
<td>12.743</td>
<td>2.77</td>
<td>3.15</td>
</tr>
<tr>
<td>C</td>
<td>55.550</td>
<td>10.155</td>
<td>5.47</td>
<td>2.51</td>
</tr>
<tr>
<td>D</td>
<td>71.653</td>
<td>8.941</td>
<td>8.01</td>
<td>2.21</td>
</tr>
<tr>
<td>E</td>
<td>96.012</td>
<td>7.727</td>
<td>12.43</td>
<td>1.91</td>
</tr>
<tr>
<td>F</td>
<td>148.565</td>
<td>6.210</td>
<td>23.92</td>
<td>1.54</td>
</tr>
</tbody>
</table>
In Figure 3 are shown the computational results for the transmission loss of the six configurations listed in Table 1. Consistent with Figure 1, a reduction in the primary resonance frequency, accompanied by quarter-wave resonances, is clearly seen as \( l/d \) increases.

5. EXPERIMENTAL APPROACH

The experimental set-up that implements the two-microphone technique, rather than the conventional SWR (Standing-Wave-Ratio) method, is shown in Figure 4. At one end of the tube a loudspeaker, driven by a signal generator module (B & K 3107), produces wide-band noise. The largest diameter in the system for any given test was in the resonator volume, with the maximum being \( d = 15.319 \) cm for the \( l/d = 1.59 \) case. A conservative limit for one-dimensional propagation in the system, corresponding to the first diametral mode in this volume, is

\[
\pi d/\lambda < 1.841, \tag{15}\]

Figure 3. The computational Helmholtz resonator transmission loss for various length/diameter ratios (geometry of Figure 1). ---, \( l/d = 1.59 \); ---, \( l/d = 2.77 \); ---, \( l/d = 5.47 \); ---, \( l/d = 8.01 \); ---, \( l/d = 12.43 \); ---, \( l/d = 23.92 \).
where $\lambda$ is the wavelength. Nothing that $\lambda f = c_0$, equation (15) may be expressed, in terms of frequency, as

$$f < \frac{1.841}{\pi} \left( \frac{c_0}{d} \right) = 0.586 \frac{c_0}{d},$$

(16)

which gives for the volume under consideration, with $c_0 = 343$ m/s, $f < 1312$ Hz as an upper frequency limit, which is substantially higher than the frequency range of interest. The upstream tube connects the loudspeaker box to the test element and the downstream tube is anechoically terminated. Four 1/4-inch condenser microphones (B & K 4135) are used in two pairs mounted upstream and downstream of the silencer flush with the interior tube surfaces. The microphones in each pair are spaced $s = 3.556$ cm apart, satisfying

$$s < \ell = \frac{1}{2} \lambda_{\text{min}} = \frac{1}{2} \frac{c_0}{f_{\text{max}}},$$

(17)

for the highest frequency of interest. Employing $f = 1312$ Hz where one-dimensional propagation breaks down yields $\ell = 13.1$ cm, indicating a spacing consistent with the bounds of equation (17). Four microphone signals were processed by a modular, multi-channel analysis system (B & K 3550), which includes a signal analyzer (B & K 2035) and a multi-channel data acquisition unit (B & K 2816), coupled with a compatible 100 kHz zoom processor (B & K 3157), 25 kHz input (B & K 3015) and interface (B & K 7521) modules. Combination of upstream and downstream auto- and cross-spectrums with the upstream and downstream distances of microphone spacing yields the transmission loss [21, 22].

6. EXPERIMENTAL RESULTS

Experimental results for transmission loss corresponding to the geometries of Table 1 are presented in Figure 5. The inverse relationship between the primary resonance frequency and $l/d$ is again observed along with secondary, length-controlled resonances. In Figure 6 are compared the primary resonance frequencies obtained from equation (4),
the numerical predictions and the experiment. The same data is also included in Table 2 along with the resonance frequencies predicted from equations (1) and (9). The theory and numerical model both assume one-dimensional flow, and correlate relatively well. Furthermore, a comparison of columns 4 and 5 of Table 2 shows that for this geometry and the frequency range considered, a lumped connector is a reasonable assumption. The analytical and computational results agree with the basic trends of the experimental data, though some deviation is observed, particularly at low \( l/d \) ratios, due to a degree of three-dimensionality in the actual phenomena. Following Ingard [16], an end correction for the local multi-dimensional effects at the volume side of the connector,

\[
\delta_v = 2d_n \sum_{n=1}^{\infty} \left( \frac{2}{b_n d_n} \right) \left[ \frac{J_1(b_n d_n/2)}{J_0(b_n d_n/2)} \right]^2
\]

may be added to the neck length, where

\[
J_1(b_n d_n/2) = 0, \quad n = 0, 1, 2, \ldots,
\]

Figure 6. Primary resonance frequencies as a function of \( l/d \). ——, Acoustic theory (equation (8)); ⋄, numerical model; ▲, experiment.
Table 2

Primary resonance frequencies (in Hz) from analytical, computational and experimental data

\(c_0 = 343.7 \text{ m/s}\)

<table>
<thead>
<tr>
<th>Resonator</th>
<th>(l/d)</th>
<th>(f_r), equation (1)</th>
<th>(f_r), equation (8)</th>
<th>(f_r), equation (9)</th>
<th>(f_{r,\text{computational}})</th>
<th>(f_{r,\text{experiment}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.59</td>
<td>100.2</td>
<td>96.7</td>
<td>97.0</td>
<td>92.0</td>
<td>89</td>
</tr>
<tr>
<td>B</td>
<td>2.77</td>
<td>100.2</td>
<td>93.5</td>
<td>93.8</td>
<td>90.5</td>
<td>87</td>
</tr>
<tr>
<td>C</td>
<td>5.47</td>
<td>100.2</td>
<td>85.6</td>
<td>85.8</td>
<td>84.0</td>
<td>81</td>
</tr>
<tr>
<td>D</td>
<td>8.01</td>
<td>100.2</td>
<td>78.6</td>
<td>78.8</td>
<td>77.6</td>
<td>75</td>
</tr>
<tr>
<td>E</td>
<td>12.43</td>
<td>100.2</td>
<td>68.3</td>
<td>68.4</td>
<td>67.8</td>
<td>66</td>
</tr>
<tr>
<td>F</td>
<td>23.92</td>
<td>100.2</td>
<td>51.0</td>
<td>51.0</td>
<td>51.0</td>
<td>50</td>
</tr>
</tbody>
</table>

and \(J_i\) is the Bessel function of the first kind of order \(i\). Note that for \(d_e/d_c \leq 0.5\), equation (18) may be closely represented by

\[
\delta_e \approx 0.425d_c(1 - 1.25d_e/d_c).
\] (20)

Combining equation (18) with equation (8) and the experimentally determined resonance frequency, an end correction may be inferred to account for the more complex physics at the main duct-connector interface. However, since a single connector geometry was used for all resonators in the study, the usefulness of these results is limited. Therefore, this aspect is not explored further in the present work.

7. CONCLUDING REMARKS

The study has shown that individual volume dimensions of Helmholtz resonators, which are neglected by classical Helmholtz theory, can have a significant effect on the resonance frequency and transmission loss characteristics. For concentric resonators, an increase in the volume length-to-diameter ratio reduces the primary resonance frequency, similar to the effect of adding a correction length to the connector. Although the analytical expression and numerical simulation employed assume one-dimensional propagation, both demonstrate well the basic behavior observed in the experiments. While the predictions from the numerical method are presented here for small amplitude disturbances, in general it is intended, provided that the one-dimensional assumption is satisfied, for applications to more complex non-linear flow phenomena where closed form solutions are not available. The relative effects of varying \(l/d\) and of multi-dimensionality are clearly dependent on the geometry of the entire resonator, including the neck dimensions as well as the volume size and shape. A multi-dimensional computational study will therefore yield results closer to those of experiments.

REFERENCES


