Wave reflections from duct terminations

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The reflection coefficients and inertial end corrections of several duct terminations, including finite length duct extensions perpendicular to an infinite wall, as well as at a number of angles, curved interface surfaces, and annular cavities, are determined and analyzed in the absence of flow by employing the boundary element method. Predictions for the classical unflanged and flanged circular ducts show good agreement with analytical and computational results available in the literature. The predictions for curved interface surfaces (bellmouth or horn) are also consistent with the available experimental data. In view of its high reflection coefficient, the duct termination with an annular cavity may be suggested for the suppression of noise radiation in a specific frequency band or for an effective wave reflection from the termination. © 2001 Acoustical Society of America. [DOI: 10.1121/1.1348298]

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I. INTRODUCTION

Part of the acoustic wave reaching a duct exit is reflected back to the source and the remainder radiates into the surroundings, presenting a coupled phenomenon between the sound propagation in the duct and external sound radiation at this boundary. The reflection and radiation characteristics depend strongly on the geometry of duct termination. Thus, the reflection coefficient amplitude and the inertial end correction (or the complex reflection coefficient) of duct terminations are the important parameters for the complete acoustic analysis of duct systems.

For an infinite circular duct without flange, Levine and Schwinger\(^1\) presented an elaborate analytical derivation for the reflection coefficient and end correction in the absence of mean flow. While the numerical results were given as a function of Helmholtz number, \(H = k a\), the limiting value of the end correction \(\delta\) was determined to be \(\delta a/\delta a_\infty = 0.6133\) as \(H \to 0\), with \(k\) being the wave number and \(a\) the duct radius. Davies et al.\(^2\) provided empirical fits to end correction of Ref. 1 as \(\delta a/\delta a_\infty = 0.6133 - 0.1168H^2\) for \(H < 0.5\) and \(\delta a/\delta a_\infty = 0.6393 - 0.1104H\) for \(0.5 < H < 2\).

For a circular duct with an infinite rigid flange, Rayleigh\(^3\) obtained an approximate value of the end correction as \(\delta a/\delta a_\infty = 0.8242\), while Daniell\(^4\) provided the bounds as \(0.8241 < \delta a/\delta a_\infty < 0.8281\) at zero frequency. This estimate was later recalculated by Nomura et al.\(^5\) as \(\delta a/\delta a_\infty = 0.8217\), by Kergomard and Garcia\(^6\) as \(\delta a/\delta a_\infty = 0.26153\pi = 0.8216\), and by Norris and Sheng\(^7\) as \(\delta a/\delta a_\infty = 0.82159\). Nomura et al.\(^3\) used Weber–Schrödinger integrals and Jacobi polynomials to derive a coupled system of two infinite sets of linear equations for the unknowns, which are then solved numerically. An approximate expression, \(\delta a/\delta a_\infty = 0.8217 - 0.367H^2\), was provided by Peters et al.\(^8\) for \(H < 0.5\). Norris and Sheng\(^7\) also presented an analytical approach for this configuration. While the method also involves numerical computation, only a single set of equations needs to be solved for the modal amplitudes in the duct. They implemented a rational function approximation to get the expressions for the reflection coefficient and end correction of the flanged and unflanged ducts. Ando\(^9\) and Peters et al.\(^3\) included the effect of duct wall thickness on the reflection coefficient and end correction. Their studies have demonstrated that, as the thickness of the duct wall is increased, these two quantities vary between the limits of the unflanged duct and infinite flanged duct at lower Helmholtz numbers. Ando’s work has been reconsidered later by Bernard and Denardo.\(^10\) Peters et al.\(^3\) also experimentally investigated the reflection coefficient and end correction of a duct terminated by a horn with a radius of curvature twice the duct diameter. Their measurements revealed \(\delta a/\delta a_\infty = 2.3\) for low \(H\), and an increasing \(\delta a/\delta a_\infty\) with increasing \(H\) for \(H < 0.3\).

While the foregoing studies have provided information for some basic configurations, there remains a significant need to determine the reflection coefficient and the end correction for a number of other duct terminations that are frequently encountered in practice. The objective of the present study is then to calculate and analyze these two quantities for a chosen set of duct terminations, including finite-length duct extensions (perpendicular as well as at an oblique angle to the wall), bellmouthed interfaces, and interfaces with annular cavities.

Following the Introduction, Sec. II provides the formulation for the calculation of reflection coefficient amplitude and inertial end correction of duct termination, and Sec. III develops a numerical approach based on the subdomain boundary element method. The results from the predictions are discussed in Sec. IV. The study is concluded with final remarks in Sec. V.

II. FORMULATION

The duct terminations considered in this study include (1) a finite duct, (2) an extended duct from an infinite rigid
To evaluate the acoustic load impedance \( Z_l \) coefficient amplitude \( u \) is developed, as described next.

The complex reflection coefficient \( R \) at the exit of duct can then be expressed in terms of \( Z_r \) as

\[
R = -|R| \exp(-j2k \delta) = \frac{Z_r - 1}{Z_r + 1},
\]

where \( \delta \) is the inertial end correction of the duct. Thus, once \( Z_l \) is determined, Eq. (2) gives \( Z_r \) and Eq. (3) the reflection coefficient amplitude \( |R| \) and the inertial end correction \( \delta \).

To evaluate the acoustic load impedance \( Z_l \) that accounts for the multidimensional waves, particularly in the neighborhood of duct exit, a subdomain boundary element approach is developed, as described next.

### III. NUMERICAL APPROACH

To employ the boundary element method in determining the acoustic load impedance of the duct, the sound field is first divided into the internal and the external acoustic domains separated by the opening surface.

#### A. Internal acoustic domain

The internal domain may also be divided into several subdomains. For the internal domain or each subdomain, the boundary integral expression is represented\(^{11}\) as

\[
C(X) P(X) = \int_{\Gamma} \left[ G(X,Y) \frac{\partial P}{\partial n}(Y) - P(Y) \frac{\partial G}{\partial n}(X,Y) \right] d\Gamma(Y),
\]

where \( \Gamma \) is the boundary surface of the acoustic domain, \( n \) is the unit normal vector on \( \Gamma \) directed away from the domain, the function \( G(X,Y) = \exp(-jkR)/4\pi R \) is Green’s function of free space, \( R \) being the distance between any two points \( X \) and \( Y \) in the domain or on the surface, and \( C(X) \) is a coefficient which depends on the position of point \( X \) and may be evaluated\(^{11}\) by

\[
C(X) = \int_{\Gamma} \frac{\partial}{\partial n} \left( \frac{1}{4\pi R} \right) d\Gamma(Y).
\]

A numerical solution of the boundary integral Eq. (4) can be achieved by discretizing the boundary surface of the domain into a number of elements. By using discretization and numerical integration, the following algebraic system of equations is obtained\(^{11}\)

\[
[F] [P] = \rho c [G] [V],
\]

where \( [F] \) and \( [G] \) are the coefficient matrices, and \( \{P\} \) and \( \{V\} \) are the vectors of sound pressure and outward normal particle velocity at boundary nodes, respectively, \( \rho c \) is the characteristic impedance of the medium. The boundaries are grouped into the inlet, opening and wall, represented by the subscripts \( i, o, \) and \( w \), respectively. Equation (6) combined with the rigid wall boundary condition yields\(^{12}\)

\[
\begin{align*}
P_i^d &= \rho c [T^d] V_o^d, & \text{(for the duct)}, \\
P_o^d &= \rho c [T^c] V_o^c, & \text{(for the cavity)}.
\end{align*}
\]

Combining duct and cavity gives

\[
\begin{align*}
P_i^d &= \rho c [T^d] V_i^d, \\
P_o^d &= \rho c [T^c] V_o^c,
\end{align*}
\]

where

\[
\begin{align*}
\{P_o^d\} &= \{P_o^c\}, \\
\{V_o^d\} &= \{V_o^c\},
\end{align*}
\]

and

\[
[T^d] = \begin{bmatrix} T^d_{ii} & 0 \\ 0 & T^d_{cc} \end{bmatrix}
\]

is the transfer impedance matrix between the inlet and opening for the internal acoustic domain.

#### B. External acoustic domain

For the full-space external sound field, the boundary integral expression is represented\(^{11}\) as

\[
C^e(X) P^e(X) = \int_{\Gamma} \left[ G(X,Y) \frac{\partial P^e}{\partial n}(Y) - P^e(Y) \frac{\partial G}{\partial n}(X,Y) \right] d\Gamma(Y),
\]

\[
C^e(X) = 1 - \int_{\Gamma} \frac{\partial}{\partial n} \left( \frac{1}{4\pi R} \right) d\Gamma(Y).
\]

Similar to the formulation of internal acoustics, discretization and numerical integration of Eq. (10) lead to the following algebraic system of equations:

\[
[F^e] [P^e] = \rho c [G^e] [V^e].
\]

The boundaries are grouped into the opening and wall, represented by subscripts \( o \) and \( w \), respectively. Equation (12) combined with the rigid wall boundary condition yields

\[
\{P_o^e\} = \rho c [T^e] \{V_o^e\},
\]

\([T^e]\) being the radiation impedance matrix for the external acoustic domain.

For the external acoustic domain with duct extension from an infinite rigid wall, the boundary integral equation can be written\(^{11}\) as
where $G_w$ is the wall of extended duct from the infinite wall, $G_o$ is the opening, $G_H = \exp(-jkr) + \frac{1}{4\pi R} \exp(-jkr_1)$ is the half-space Green’s function, $R_1$ being the distance between $Y$ and the image point of $X$ with respect to the flange; and

$$C'(X) = 1 - \int_{\Gamma_0} \left( \frac{1}{4\pi R} \right) d\Gamma(Y),$$

(15)

where $\Gamma_0$ is the opening, $\Gamma$ being the total boundary of the duct extended from the infinite wall. Similar to the full-space problem, the acoustic radiation impedance matrix $[T^e]$ for this case can be obtained by using discretization and numerical integration of Eq. (14). For the limiting case of no-extension (a duct with infinite rigid flange), the sound pressure at the opening may be expressed as

$$P^e(X) = \int_{\Gamma_0} \left[ \frac{1}{4\pi R} \frac{\partial P^e}{\partial n}(Y) \right] d\Gamma(Y).$$

(16)

Similarly, the acoustic radiation impedance matrix $[T^e]$ for the semi-infinite external acoustic domain can also be obtained by using discretization and numerical integration of Eq. (16).

**C. Coupling**

At the opening, the solution should satisfy the continuity conditions of sound pressure and particle velocity:

$$\{P^t_o\} = \{P^t_i\},$$

(17)

$$\{V^t_o\} = -\{V^t_i\}.$$  

(18)

Combining Eqs. (9) and (13) with the continuity conditions (17) and (18) yields the acoustic load impedance as

$$Z_l = [T_{11} - T_{12}(T_{22} + T^e)^{-1}T_{22}].$$

(19)
IV. RESULTS AND DISCUSSION

First, the effect of length of an unflanged finite duct (Fig. 1) on the reflection coefficient and end correction is examined. Figure 2 shows the numerical predictions of reflection coefficient and end correction for four different duct lengths \( l/a = 0.5, 1.0, 2.0, 3.0 \). For a duct with \( l/a > 2 \), the results illustrate that the effect of duct length on these two quantities becomes negligible in the region of \( H < 3 \), and the present predictions agree with the analytical results of Levine and Schwinger\(^1\) for the classical configuration of the infinite duct without flange.

The effect of duct extension \( l_e \) from the infinite rigid wall (Fig. 3) on the reflection coefficient and end correction is shown in Fig. 4. The finite extension introduces an oscillatory behavior on both quantities as a function of \( H \) due to the presence of an infinite rigid wall. The period of oscillations in \( H \) may be crudely related to two subsequent quarter wave resonances between the duct opening and the infinite rigid wall leading to \( \Delta H \approx \pi l_e (l_e + \delta)/a \). As expected, the oscillatory behavior diminishes as \( l_e/a \) becomes large. For the limiting case of no-extension \( (l_e/a = 0, \text{flanged duct}) \), the present predictions of the reflection coefficient and end correction are in good agreement with the available literature (Rayleigh,\(^3\) Daniell,\(^4\)

FIG. 4. Duct extended from an infinite rigid flange: (a) reflection coefficient and (b) end correction.

FIG. 5. Duct at an angle \( \theta = 45^\circ \) extended from an infinite rigid flange.

FIG. 6. Duct at an angle \( \theta = 45^\circ \) extended from an infinite rigid flange: (a) reflection coefficient and (b) end correction.
Nomura et al., ⁵ Kergomard and Garcia, ⁶ and Norris and Sheng ⁷. The present predictions in Fig. 4 for \( \frac{l_e}{a} = 0 \) agree very well with Fig. 7 of Nomura et al. ⁵ to the degree that the two sets of results cannot be distinguished from each other over \( 0 < H < 3.0 \), including, for example, a specific value of \( \frac{\delta}{a} = 0.821 \) for \( H = 0.05 \).

The predictions for the reflection coefficient and end correction for a duct extension from an infinite rigid wall with an oblique angle of \( \theta \) (Fig. 5) are shown in Fig. 6 for \( \theta = 45^\circ \). Similar to the duct extension with \( 90^\circ \) angle, the oscillatory behavior is observed again. Figures 7 and 8 compare the reflection coefficients and end corrections for, respectively, the unflanged and flanged ducts with \( l_e = 0 \) and \( \theta = 30^\circ, 45^\circ, 60^\circ, 90^\circ \). At higher Helmholtz numbers ducts with larger \( \theta \) provide, in general, higher reflection coefficients. In view of the definition of \( \delta \) in Fig. 5, a reduction on the order of radius may be anticipated for \( \theta = 45^\circ \) (relative to \( \theta = 90^\circ \)) at low Helmholtz numbers as shown in Fig. 7. Note that \( H = 1.8 \) in Figs. 6–8 to remain below the first diametral mode (1,0).

The acoustic characteristics of a duct with bellmouth (Fig. 9) are examined next to understand the effect of radius of curvature on the reflection coefficient and end correction. The latter is defined relative to the end of the straight section of the duct, consistent with Peters et al. ⁸ Figures 10 and 11 depict the computed results for these two quantities for five different radii of curvature (\( r/a = 0.5, 1.0, 2.0, 3.0, 4.0 \)), and without and with an infinite flange, respectively. With increasing radius of curvature, the reflection coefficient decreases and the end correction increases. For the special case of \( r/a = 4 \), the present predictions agree with the experimental results of Peters et al. ⁸ who determined \( \delta/a \approx 2.3 \) at low values of \( H \). For this special case at \( H = 0.025 \), for example, the predicted numerical values are \( \delta/a = 2.322 \) and

![FIG. 7. Unflanged ducts with \( \theta = 30^\circ, 45^\circ, 60^\circ, 90^\circ \): Comparison of (a) reflection coefficient and (b) end correction.](image1)

![FIG. 8. Flanged ducts with \( l_e = 0 \) and \( \theta = 30^\circ, 45^\circ, 60^\circ, 90^\circ \): Comparison of (a) reflection coefficient and (b) end correction.](image2)

![FIG. 9. Ducts with unflanged and flanged bellmouth.](image3)
\( \delta/a = 2.369 \) for Fig. 10(b) and Fig. 11(b), respectively. The differences in these configurations beyond the radiused inlet do not have a significant impact on the end correction for low \( H \). In comparison with the straight duct, the duct with bellmouth leads to much lower wave reflection at higher Helmholtz numbers. Thus the bellmouth with large radius of curvature should be avoided, for example, at the exhaust exit in order to promote effective pressure reflections from the termination for noise control. Figure 12 shows the effect of curved interface with the radiused length subtracted from the end correction, transforming the dimensionless end correction to \( (\delta - r)/a \). Such manipulation essentially shifts the reference point for end correction to the very end of the duct (end of bellmouth section). Relative to the end of the curved duct, Fig. 12 then suggests that a duct with a large-radius termination, such as \( r/a = 4 \), will appear to be shorter than the actual total length when the end correction is incorporated.

Finally, ducts with annular cavities (unflanged and flanged) are considered as shown in Fig. 13 with a constant cross section and Fig. 14 with a step transition. The computed results for the reflection coefficient and the end correction are compared in Fig. 15 for \( l_p/c = 4 \) and \( c/a = 1/2 \). The results with cavities here may be contrasted with the ducts alone of Figs. 2 and 4. As a result of the interaction of waves reflected from attached cavities with those radiated from the opening, it is important to note that (1) the behavior of reflection coefficient changes dramatically including the additional peaks, therefore the sound radiation may be suppressed effectively in a specific frequency band, and/or (2) the pressure wave reflections from the termination back to the duct would be strengthened. Compared to the uniform cavity, the step cavity moves the first additional peak to a higher Helmholtz number, and the second peak to a lower Helmholtz number. The high reflection coefficient, in a specific frequency band, of duct with annular cavity may be used to compensate the low attenuation in the corresponding frequency region of a silencer, therefore an improved acoustic attenuation performance may be obtained. Similar behavior can possibly be utilized to promote stronger pressure wave reflections from the opening, thereby providing mechanisms to manipulate engine tuning.

V. CONCLUDING REMARKS

This study has shown the effect of duct termination geometry on the reflection coefficient and the inertial end correction by employing the subdomain boundary element approach. These two quantities were determined and analyzed for (1) an unflanged finite duct, (2) a duct extended from an infinite rigid wall, (3) a duct extended with an angle \( \theta \) from
an infinite rigid wall, (4) ducts with unflanged and flanged bellmouth, and (5) ducts with unflanged and flanged annular cavities. The effect of flow, which was well established in the literature for certain configurations (see, for example, Peters et al.), as a function of Mach and Strouhal numbers was excluded from the study. Instead, the current work has chosen to concentrate on quantifying the impact of interface geometry which has not been investigated in detail with the exception of limiting cases. For unflanged configurations, a finite duct with \( l/a > 2 \) exhibits similar acoustic characteristics to an infinite duct at low Helmholtz numbers. The duct extension from an infinite rigid wall introduces an oscillatory behavior in the reflection coefficient and end correction near the baseline unflanged duct due to the interaction of waves reflected from the infinite wall and those radiated from the duct opening. Within about a diameter-distance from the

FIG. 12. \((\delta - r)/a\) for ducts with (a) unflanged bellmouth and (b) flanged bellmouth.

FIG. 13. Ducts with unflanged and flanged annular uniform cavity.


FIG. 15. Ducts with unflanged and flanged annular cavity with \( l/c = 4 \) and \( c/a = 1/2 \): (a) reflection coefficient and (b) end correction.
wall, the acoustic behavior of the two quantities approaches that of an unflanged wall with the exception of oscillations. The oblique duct extension from an infinite rigid wall, in general, decreases the reflection coefficient of a duct at higher Helmholtz numbers as compared to the perpendicular extension. The bellmouth interface reduces the reflection coefficient significantly at higher Helmholtz numbers. The duct termination with cavity influences end correction and introduces additional peaks in the reflection coefficient and may be used for the suppression of noise radiation, or to promote effective wave reflection from the open end.

3 Lord Rayleigh, The Theory of Sound (Dover, New York, 1945).